1) a) $X(u,v) = (a\sin u\cos v,\ b\sin u\sin v,\ c\cos u)$

$X_u = (a\cos u\cos v,\ b\cos u\sin v,\ -c\sin u)$

$X_v = (-a\sin u\sin v,\ b\sin u\cos v,\ 0)$

$E = \langle X_u, X_u\rangle = a^2\cos^2 u\cos^2 v + b^2\cos^2 u\sin^2 v + c^2\sin^2 u$

$= (a^2+b^2)\cos^2 u + c^2\sin^2 u$

$F = \langle X_u, X_v\rangle = -a^2\cos u\sin u\cos v\sin v + b^2\cos u\sin u\cos v\sin v$

$= (b^2 - a^2)\cos u\sin u\cos v\sin v.$

$G = \langle X_v, X_v\rangle = a^2\sin^2 u\sin^2 v + b^2\sin^2 u\cos^2 v$

b) $X(u,v) = (au\cosh v,\ bu\sinh v,\ u^2)$

$X_u = (a\cosh v,\ b\sinh v,\ 2u)$

$X_v = (au\sinh v,\ bu\cosh v,\ 0).$

$E = \langle X_u, X_u\rangle = a^2\cosh^2 v + b^2\sinh^2 v + 4u^2$

$F = \langle X_u, X_v\rangle = a^2u\cosh v\sinh v + b^2u\cosh v\sinh v$

$G = \langle X_v, X_v\rangle = a^2u^2\sinh^2 v + b^2u^2\cosh^2 v$

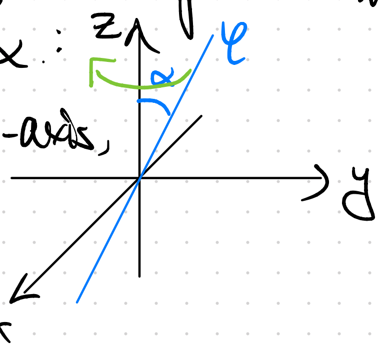
2) We first show that $X(u,v) = (u\sin\alpha\cos v, u\sin\alpha\sin v, u\cos\alpha)$ is a param. of the cone with angle of vertex 2α.

Note that $X(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$

where $(f(u), g(u)) = (u\sin\alpha, u\cos\alpha)$

i.e. it is a surface of revolution of the curve $\varphi(u) = (f(u), g(u))$ around z-axis.

We now show the curve $\varphi(u)$ is a straight line through the origin with internal angle α:



So when we revolve around z-axis, the angle at vertex will be 2α.

Note that

$\varphi(0) = (0,0)$, so φ passes through origin.

Internal angle θ:

$$\tan\theta = \frac{f(u)}{g(u)} = \frac{\cancel{u}\sin\alpha}{\cancel{u}\cos\alpha} = \tan\alpha \Rightarrow \theta = \alpha.$$

So X is the param. of cone with angle at vertex $= 2\alpha$.

2^{nd} Part: For convenience, we'll first compute E, F, G of this param.

$X_u = (\sin\alpha\cos v, \sin\alpha\sin v, \cos\alpha)$

$X_v = (-u\sin\alpha\sin v, u\sin\alpha\cos v, 0)$

$$E = \langle X_u, X_u \rangle = \sin^2\alpha\cos^2 v + \sin^2\alpha\sin^2 v + \cos^2\alpha = \sin^2\alpha + \cos^2\alpha = 1.$$

$$F = \langle X_u, X_v \rangle = -u\sin^2\alpha\cos v\sin\alpha + u\sin^2\alpha\cos v\sin v = 0.$$

$$G = \langle X_v, X_v \rangle = u^2\sin^2\alpha\sin^2 v + u^2\sin^2\alpha\cos^2 v = u^2\sin^2\alpha.$$

Now let $\psi(t)$ be a generator of the cone, ie.

$$\psi(t) = X(t, t_0), \quad t_0 = \text{const}.$$

Then the angle to be calculated, θ, is given by

$$\cos\theta = \frac{\langle \gamma'(t), \psi'(t) \rangle}{|\gamma'(t)||\psi'(t)|}.$$

By chain rule, $\psi'(t) = (t)'X_u + \overset{0}{\cancel{(t_0)'}}X_v = X_u$

$$|\psi'(t)| = |X_u| = 1 \text{ by above.}$$

Since $\gamma(t) = X(\underbrace{c\exp(t\sin\alpha\cot\beta)}_{u(t).}, t)$,

then $\gamma(t) = X(u(t), t)$ and by chain rule, we have

$$\gamma(t) = u'(t)X_u + X_v$$

$$u'(t) = c\exp(t\sin\alpha\cot\beta)\sin\alpha\cot\beta = \sin\alpha\cot\beta\, u(t)$$

$$|\gamma'(t)|^2 = \langle \gamma'(t), \gamma'(t) \rangle = \langle u'(t)X_u + X_v, u'(t)X_u + X_v \rangle$$

$$= \sin^2\alpha\cot^2\beta\, u^2(t)E + 2u'(t)\cancel{F}^{0} + G$$

$$= \sin^2\alpha\cot^2\beta\, u^2(t) + u^2(t)\sin^2\alpha$$

$$= u^2(t)\sin^2\alpha(1+\cot^2\beta).$$

$\Rightarrow |\gamma'(t)| = u(t)\sin\alpha\,(1+\cot^2\beta)^{1/2}$

So $\cos\theta = \dfrac{\langle \gamma'(t), \psi'(t)\rangle}{|\gamma'(t)|\,|\cancel{\psi'(t)}|^{1}}$

$$= \frac{\langle \sin\alpha\cot\beta\, u(t) X_u + X_v ,\ X_u\rangle}{u(t)\sin\alpha\,(1+\cot^2\beta)^{1/2}}$$

$$= \frac{\cancel{\sin\alpha}\cot\beta\,\cancel{u(t)}}{\cancel{u(t)}\,\cancel{\sin\alpha}\,(1+\cot^2\beta)^{1/2}}$$

$$= \frac{\cot\beta}{(1+\cot^2\beta)^{1/2}} = \frac{\cot\beta}{\csc\beta} = \frac{\cos\beta}{\sin\beta}\Big/\frac{1}{\sin\beta} = \cos\beta.$$

$\Rightarrow \cos\theta = \cos\beta$. So the angle of intersection is β. //

3) a) Let $X: U \subseteq \mathbb{R}^2 \to S$ be a parameterization, $p = X(u,v)$.

We want to write $\operatorname{grad}(f)_p = \alpha X_u + \beta X_v$ for some α, β. We want to find α, β.

In particular,

$$\langle \operatorname{grad}(f)_p, X_u \rangle = \langle \alpha X_u + \beta X_v, X_u \rangle = \alpha \langle X_u, X_u \rangle + \beta \langle X_v, X_u \rangle$$
$$= \alpha E + \beta F.$$

By def'n of $\operatorname{grad}(f)_p$, $LHS = df_p(X_u) = \begin{bmatrix} f_u & f_v \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = f_u$.

$\Rightarrow f_u = \alpha E + \beta F$.

Similarly, $\langle \operatorname{grad}(f)_p, X_v \rangle = \alpha F + \beta G$

and $f_v = df_p(X_v) = \langle \operatorname{grad}(f)_p, X_v \rangle = \alpha F + \beta G$.

So we have the system

$$\begin{cases} f_u = \alpha E + \beta F \\ f_v = \alpha F + \beta G \end{cases} \quad \text{i.e.} \quad \begin{bmatrix} E & F \\ F & G \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} f_u \\ f_v \end{bmatrix}.$$

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} = \frac{1}{EG - F^2}\begin{bmatrix} G & -F \\ -F & E \end{bmatrix} \Rightarrow \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{1}{EG - F^2}\begin{bmatrix} G & -F \\ -F & E \end{bmatrix}\begin{bmatrix} f_u \\ f_v \end{bmatrix}$$

$$\Rightarrow \alpha = \frac{f_u G - f_v F}{EG - F^2}, \quad \beta = \frac{f_v E - f_u F}{EG - F^2}.$$

So $$\operatorname{grad}(f)_p = \frac{f_u G - f_v F}{EG - F^2} X_u + \frac{f_v E - f_u F}{EG - F^2} X_v.$$

b) By def'n, $df_p(v) = \langle \operatorname{grad}(f)_p, v \rangle$

$$\leq |\langle \operatorname{grad}(f)_p, v \rangle|$$

Cauchy Schwarz $\longrightarrow$ $\leq |\operatorname{grad}(f)_p| \underbrace{|v|}_{1} = |\operatorname{grad}(f)_p|$.

and maximum is attained iff v and $\operatorname{grad}(f)_p$ are linearly dependent, i.e. $v = \lambda \operatorname{grad}(f)_p$ for some $\lambda \in \mathbb{R}$.

Since $|v| = 1$, we must have $\lambda = \pm \frac{1}{|\operatorname{grad}(f)_p|}$.

Reinserting into ineq. above, we see that $\lambda = \frac{1}{|\operatorname{grad}(f)_p|}$ for first inequality to be an equality.

i.e. $df_p(v)$ achieves max among $|v| = 1$ iff

$$v = \frac{\operatorname{grad}(f)_p}{|\operatorname{grad}(f)_p|}.$$

4) At p, we decompose T_pS into the eigenspaces of S_p, ie.

$S_p = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$, k_1, k_2 are principal curvatures/eigenvalues

and u_1, u_2 are the principal directions/eigenvectors.

Since $H=0$, $\frac{1}{2}(k_1+k_2)=0 \Rightarrow k_2=-k_1$ and

$K_p = k_1k_2 = -k_1^2$.

Then write $v=a_1u_1+a_2u_2$, $w=b_1u_1+b_2u_2$.

We have

$$\langle dN_p(v), dN_p(w)\rangle = \langle -S_p(v), -S_p(w)\rangle$$

$$= \langle S_p(a_1u_1+a_2u_2), S_p(b_1u_1+b_2u_2)\rangle$$

$$= \langle a_1S_p(u_1)+a_2S_p(u_2), b_1S_p(u_1)+b_2S_p(u_2)\rangle$$

$$= \langle a_1k_1u_1 + a_2k_2u_2, b_1k_1u_1+b_2k_2u_2\rangle.$$

$$= \langle a_1k_1u_1 - a_2k_1u_2, b_1k_1u_1 - b_2k_1u_2\rangle$$

$$= k_1^2\langle a_1u_1 - a_2u_2, b_1u_1-b_2u_2\rangle$$

$$= k_1^2\Big(a_1b_1\overset{1}{\langle u_1,u_1\rangle} - a_1b_2\overset{0}{\langle u_1,u_2\rangle} - a_2b_1\overset{0}{\langle u_2,u_1\rangle}$$
$$+ a_2b_2\overset{1}{\langle u_2,u_2\rangle}\Big)$$

$$= -K_p\langle v,w\rangle.$$

Since $\langle v,w\rangle = \langle a_1u_1+a_2u_2, b_1u_1+b_2u_2\rangle = a_1b_1+a_2b_2$. //.